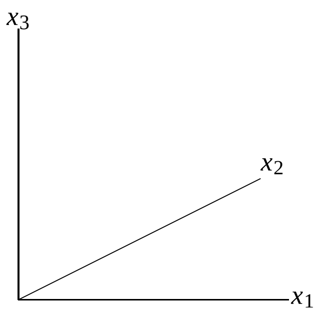
## A.4 A short note on tensor notation and coordinates

In order to save writing, and to have a better overview of the equations, we can use tensor notation. Section D.5 includes a short glossary of coordinate systems.

### Cartesian tensor notation

The equations of this book, and in many others, are written in *Cartesian tensor notation*. The reason is that most equations get a simpler outlook in this way. We start out with a three-dimensional Cartesian coordinate system:




A vector has three components, for instance a space vector and a velocity vector:
Vector form: $\vec{x}$, component form: $[x_1, x_2, x_3]$ or $[x, y, z]$.
Vector form: $\vec{u}$, component form: $[u_1, u_2, u_3]$ or $[u, v, w]$ or $[u_x, u_y, u_z]$.

The stress tensor has nine components and can be written as

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \quad \text{or} \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{A.106}$$

In Cartesian tensor form, a vector (1st order tensor) is denoted with one index, for instance

$$x_i \ \text{ with } i = 1, 2 \text{ or } 3; \qquad x_i = [x_1, x_2, x_3],$$

$$u_i \ \text{ with } i = 1, 2 \text{ or } 3; \qquad u_i = [u_1, u_2, u_3].$$

The stress tensor is a 2nd order tensor and get two indices:

$$\tau_{ij} \ \text{ with } i = 1, 2 \text{ or } 3 \text{ and } j = 1, 2 \text{ or } 3.$$

A second-order tensor, like $\tau_{ij}$, has 9 ($3^2$) components in a three-dimensional room. A third-order tensor, $a_{ijk}$, has 27 ($3^3$) components, a fourth-order tensor, $b_{ijkm}$, has 81 ($3^4$), and so on. We rarely encounter tensors with order three or higher.

The indices are chosen arbitrarily, so that $u_i$ is the same as $u_j$ or $u_n$ – that is, provided that the indices $i$, $j$ and $n$ are not used elsewhere in the term in question. Similarly, $\tau_{ij}$ is the same as $\tau_{ik}$, $\tau_{jk}$ or $\tau_{pn}$. Each of the indices has the value 1, 2 or 3.

Example: In general, $\tau_{13} \neq \tau_{23}$. These are two (of nine) individual components of the tensor. Nevertheless, we have $\tau_{ij} = \tau_{ik} = \tau_{pn}$. In the latter instance, each of the three symbols represents the *entire* tensor (all the 9 components), and it is one and the same tensor. If we in addition say that $(i, j) = (1, 3)$ and $(p, n) = (2, 3)$, it is then obvious that $\tau_{ij} \neq \tau_{pn}$. In that case, we are back to talking about individual components of the tensor.

When two indices in a tensor or a term are the same, we should sum over these indices from 1 to 3, for instance

$$u_i u_i \text{ should be read as } \sum_{i=1}^{3} u_i u_i = u_1 u_1 + u_2 u_2 + u_3 u_3,$$

$$\frac{\partial u_i}{\partial x_i} \text{ should be read as } \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3},$$

$$u_j \frac{\partial u_i}{\partial x_j} \text{ should be read as } \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3}.$$

This is known as *Einstein's summation rule*.

The last example has two $j$-s but only one $i$. The single index should not be summed. The sum on the right-hand-side represents 3 different sums, one or each value of $i$. We recognize the convection term of the momentum equation. This can be regarded as three equations, one for each of the components. In the momentum equation, the index for the component (here $u_i$) is the same in all terms of the equation. However, indices to be summed over can vary from term to term. When the convection term is $u_j(\partial u_i/\partial x_j)$, the stress term can be written as $\partial \tau_{ij}/\partial x_j$ or $\partial \tau_{ik}/\partial x_k$, but not as $\partial \tau_{ki}/\partial x_i$ or $\partial \tau_{ik}/\partial x_i$. It is more orderly when we use the same index for summing in different terms.

More examples:

$$u_i u_i = u_j u_j, \quad \text{but generally} \quad u_i u_i \neq u_i u_j,$$
$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_n}{\partial x_n}, \quad \text{but generally} \quad \frac{\partial u_i}{\partial x_i} \neq \frac{\partial u_k}{\partial x_j}.$$

The expressions to the right are unequal since one term has one index that is repeated, and hence should be summed, whereas the other have two different indices and should not be summed. The former represent one single quantity each, which is the sum of three terms. Each of the latter represent 9 different combinations of the two indices, that is, 9 different quantities.

In a term of an equation it does not matter whether it reads $\partial u_i/\partial x_i$ or $\partial u_n/\partial x_n$, that is, provided the indices $i$ and $n$ are not used otherwise in the term.

Some equations contain terms with two or more pairs of repeated indices. Then you go on summing. The product $u_i u_i$ has 3 terms that should be summed, $\tau_{ij}(\partial u_i/\partial x_j)$ has $3^2$=9 terms, $a_{ij} b_{jk} c_{ik}$ has

$3^3$=27 terms to be summed, and so on.

Two special tensors are the *Kronecker delta* $\delta_{ij}$ and the *permutation tensor* $\epsilon_{ijk}$:

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases} \qquad \text{or} \quad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A.107}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{when } (i, j, k) \text{ is (1,2,3), (2,3,1) or (3,1,2)}, \\ -1 & \text{when } (i, j, k) \text{ is (3,2,1), (2,1,3) or (1,3,2)}, \\ 0 & \text{when } i = j \text{ and/or } i = k \text{ and/or } j = k. \end{cases} \tag{A.108}$$

For two-dimensional cases, the indices get the values 1 or 2, rather than 1, 2 or 3. For one-dimensional cases all indices get the value 1.

This section is on Cartesian tensors. It is common to present the equations on Cartesian form, also when using non-Cartesian coordinate systems.

Tensor analysis, both Cartesian and general, is further described in, for instance, Tyldesley (1975), Aris (1962), Irgens (1982) and Sokolnikoff (1964). The former two are the easier to digest, and are also more related to fluid mechanics, whereas the latter two have a more general aim.

**Cylinder coordinates, spherical coordinates**

Certain forms of the fundamental equations are presented in cylinder coordinates (polar coordinates) and in spherical coordinates by Bird, Stewart and Lightfoot (1960:83,317) (also reproduced by Kuo, 1986:175–201).